Optimal Tracking Laws

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1. Introduction

TRACKING is of interest in intercept systems since intercept will occur if tracking is maintained in all but one coordinate and intercept (collision) occurs on the remaining coordinate. For example, in a line-of-sight or beam-rider system, the missile is controlled to track the line between a point of control and the target, i.e., the missile tracks in azimuth and elevation. Boost provides the closing velocity in range for intercept. In addition, the point of control must sense and track the target and possibly the missile.

Previously, the method of solution of an inhomogeneous linear-quadratic optimization problem[1,2] was applied to minimizing a terminal condition,[1,3] that is, intercept. In this Note, it will be applied to the regulator problem, that is, tracking.

2. An Inhomogeneous Linear-Quadratic Optimization Problem[1,2]

The control $u(t)$ which minimizes the quadratic performance index,

$$J(t_f) = \tfrac{1}{2}(x^T S_f x)_{t=t_f} + \tfrac{1}{2}\int_{t_0}^{t_f} (x^T A x + u^T B u)dt, \qquad (1)$$

subject to the constraints

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + c(t), \qquad x(t_0) \text{ given} \qquad (2)$$

is

$$u(t) = -B^{-1}G^T[Sx(t) + k(t)] \qquad (3)$$

where

$$\dot{S} = -SF - F^T S + SGB^{-1}G^T S - A, \qquad S(t_f) = S_f \qquad (4)$$

and

$$\dot{k} = (SGB^{-1}G^T - F^T)k - Sc, \qquad k(t_f) = 0 \qquad (5)$$

The $k(t)$ term in Eq. (3) results from the inhomogenity term $c(t)$ in Eq. (2).

In studying the intercept problem, it was assumed that the matrix A was zero.[1,3] In studying tracking, it will be assumed that $S_f = 0$.

3. Optimal Tracking—Velocity (Impulse) Control

Consider a performance index of the form

$$J(t_f) = \frac{1}{2}\int_{t_0}^{t_f} \left(\frac{x}{T}\right)^2 + \Delta v^2 \, dt \qquad (6)$$

subject to a constraint of the form

$$\dot{x}(t) = \Delta v(t) + v(t) \qquad (7)$$

Equation (4) becomes

$$\dot{S}(t) = S^2(t) - (1/T^2), \qquad S(t_f) = 0 \qquad (8)$$

The solution to this Ricatti equation is

$$S(t) = (1/T)\tanh[(t_f - t)/T] \qquad (9)$$

The contribution to the control due to the inhomogenity remains to be determined. Equation (5) becomes

$$\dot{k}(t) = (1/T)\tanh[(t_f - t)/T](k(t) - v(t)), \qquad k(t_f) = 0 \qquad (10)$$

The solution in terms of the higher derivatives of the relative motion is

$$k(t) = \sum_{n=0}^{\infty} T^{2n}\left[x^{(2n+1)}(t) - x^{(2n+1)}(t_f)\operatorname{sech}\left(\frac{t_f - t}{T}\right)\right] + \sum_{n=1}^{\infty} T^{2n-1}x^{(2n)}(t)\tanh\left(\frac{t_f - t}{T}\right) \qquad (11)$$

The control, Eq. (3), becomes

$$\Delta v(t) = (-x(t)/T)\tanh[(t_f - t)/T] - k(t) \qquad (12)$$

Note that the hyperbolic functions just reduce the control to zero at the final time, because the control has insufficient time to be integrated and reduce the error. Letting the final time approach infinity eliminates the hyperbolic functions and the need to know the derivatives of the motion at the final time. In the limit, Eq. (12) becomes

$$\Delta v(t) = \frac{-1}{T}\sum_{n=0}^{\infty} x^{(n)}(t)T^n \qquad (13)$$

For a finite final time, Eq. (13) would be suboptimal but the simplicity of control, upon which it is difficult to place cost, is desirable and may outweigh optimality in the sense that Eq. (6) is a minimum.

4. Optimal Tracking—Acceleration Control

Consider a performance index and constraint of the form

$$J(t_f) = \frac{1}{2}\int_{t_0}^{t_f} \left(\frac{2x(t)}{T^2}\right)^2 + \Delta a^2(t)dt \qquad (14)$$

and

$$\begin{pmatrix}\dot{v}(t)\\ \dot{x}(t)\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}v(t)\\ x(t)\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Delta a(t)\\ 0\end{pmatrix} + \begin{pmatrix}a(t)\\ 0\end{pmatrix} \qquad (15)$$

That is

$$A = \begin{pmatrix}0 & 0\\ 0 & \dfrac{4}{T^4}\end{pmatrix}, \qquad B = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and

$$F = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \qquad G = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

Equation (4) becomes

$$\begin{pmatrix}\dot{S}_{11}(t) & \dot{S}_{12}(t)\\ \dot{S}_{21}(t) & \dot{S}_{22}(t)\end{pmatrix} = \begin{pmatrix}S_{11}^2(t) - (S_{12}(t) + S_{21}(t)), & S_{11}(t)S_{12}(t) - S_{22}(t)\\ S_{11}(t)S_{21}(t) - S_{22}(t), & S_{12}(t)S_{21}(t) - \dfrac{4}{T^4}\end{pmatrix} \qquad (16)$$

Figure 1 is a computer plot for $T = 1$ backwards in time from

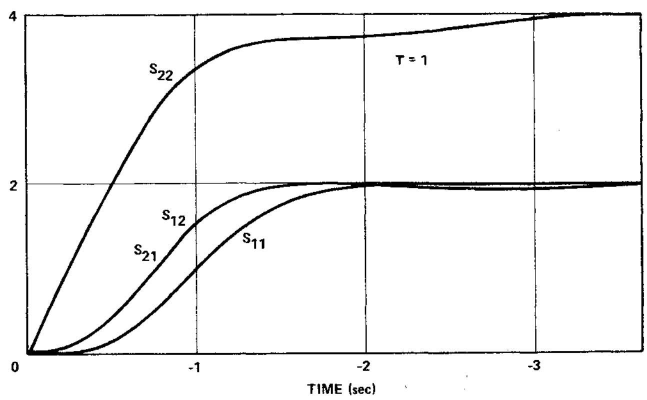

Fig. 1 Analog computer solution of Eq. (16).

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the final condition

$$S(t_f) = 0$$

Note that very quickly

$$\dot{S} \simeq 0$$

Assuming[2,4] $S = 0$, Eq. (16) may be simply solved

$$S = \begin{pmatrix} \frac{2}{T}, & \frac{2}{T^2} \\ \frac{2}{T^2}, & \frac{4}{T^3} \end{pmatrix} \tag{17}$$

Equation (5) becomes

$$\begin{pmatrix} \dot{k}_1(t) \\ \dot{k}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{T}, & -1 \\ \frac{2}{T^2}, & 0 \end{pmatrix} \begin{pmatrix} k_1(t) - a(t) \\ k_2(t) \end{pmatrix} \tag{18}$$

Forming one second-order differential equation from the two coupled first-order equations yields

$$\ddot{k}_1(t) = (2/T)[\dot{k}_1(t) - a(t)] - (2/T^2)[k_1(t) - a(t)] \tag{19}$$

The solution in terms of the higher derivatives of the relative motion is,

$$k_1(t) = \sum_{n=0}^{\infty} [x^{(n+2)}(t) - Tx^{(n+3)}(t)]T^n \sum_{N=0}^{n} \frac{i^N}{(1+i)^n} \tag{20}$$

The control law, Eq. (3), becomes

$$\Delta a = (-2/T^2)x(t) - (2/T)v(t) - k_1(t) \tag{21}$$

It is interesting to note that the jerk term $x^{(3)}$ vanishes.

5. A Conjecture

Generally, all the higher derivatives of the relative motion are not known and the control laws would be truncated. Truncating at the order of the control, Eqs. (13) and (21) become

$$\Delta v(t) = (-1/T)[x(t) + Tv(t)] \tag{22}$$

and

$$\Delta a(t) = (-2/T^2)[x(t) + Tv(t) + (T^2/2)a(t)] \tag{23}$$

The truncated intercept laws[5,3] are

$$\Delta v(t) = [-1/(t_f - t)][x(t) + (t_f - t)v(t)] \tag{24}$$

and

$$\Delta a(t) = [-3/(t_f - t)^2]\{x(t) + (t_f - t)v(t) + [(t_f - t)^2/2]a(t)\} \tag{25}$$

In comparing these laws, it becomes apparent that Eqs. (22) and (23) are continuously trying to "intercept" at some fixed time increment, T, in the future. The possibility appears for a controller which attempts to track when supplied a "time constant" and to intercept when supplied the time to go. Does man behave in such a manner?[4,6]

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Evaluation of the Averaged Specular Component of Reflectance

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Introduction

RELIABLE predictions of radiant heat transfer require acceptable models for the radiation characteristics of surfaces. Since detailed knowledge of the radiation surface properties is generally not available for engineering materials, the analytical methods conventionally used for treating radiant heat exchange assume that the surfaces are either diffuse, specular or have diffuse plus specular components of reflectance.[1] In spite of the wide acceptance of the more realistic diffuse + specular model, a method for evaluating the specular component of reflectance in terms of known variables such as temperatures, reflectances, surface roughness and configuration has not been suggested. Two very simple approximate procedures[2,3] used in the past do not appear to be either adequate or in general correct.

In this Note, rigorous definitions of local as well as over-all (averaged over a surface) specular components of reflectance are presented. Approximations are then made to reduce the definitions to a workable form from which the over-all specular component of reflectance can be calculated and some illustrative results are also presented which examine the influence of various parameters.

Local Specular Component of Reflectance

The reflectance in general varies with location on the surface since the incident radiation field $I_{i\lambda}(\mathbf{r}, \theta', \phi')$ is a function of position vector $\mathbf{r}$ and incident direction θ', ϕ'. For an isotropic surface, the ratio of specular reflectance $\rho_\lambda{}^s$ to reflectance ρ_λ on the spectral basis is defined as

$$\frac{\rho_\lambda{}^s(\mathbf{r})}{\rho_\lambda(\mathbf{r})} = \frac{\int_{\phi_1'}^{\phi_2'} \int_{\theta_1'}^{\theta_2'} \rho_\lambda{}^s(\theta') I_{i\lambda}(\mathbf{r}, \theta', \phi') \cos\theta' \sin\theta' \, d\theta' \, d\phi'}{\int_{\phi_1'}^{\phi_2'} \int_{\theta_1'}^{\theta_2'} \rho_\lambda(\theta') I_{i\lambda}(\mathbf{r}, \theta', \phi') \cos\theta' \sin\theta' \, d\theta' \, d\phi'} \tag{1}$$

If the incident radiation field $I_{i\lambda}$ and the directional reflectances are known, Eq. (1) can readily be evaluated.

The importance of various parameters on $\rho_\lambda{}^s/\rho_\lambda$ is examined by assuming that the specular component of directional reflectance can be approximated by the Beckmann model[4]

$$\rho_\lambda{}^s(\theta', \sigma/\lambda) \simeq \rho_\lambda(\theta') g(\theta', \sigma/\lambda) = \rho_\lambda(\theta') \exp\{-[4\pi(\sigma/\lambda)\cos\theta']^2\} \tag{2}$$

where $\rho_\lambda(\theta')$ is the directional reflectivity of a smooth material with finite conductivity and σ is the rms roughness. Correcting for finite conductivity in this manner does not introduce any appreciable error and is justified on the basis of experimental evidence.[5] For a special case when $I_{i\lambda}$ and ρ_λ (or material of

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